

# Form Factors for Semi-Leptonic and Radiative Decays of Heavy Mesons to Light Mesons

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To know and understand form factors of hadronic currents is of decisive importance for analysing exclusive weak decays. The ratios of different form factors of a given process depend on the relativistic spin structure of initial and final particles. It is shown — assuming simple properties of the spectator particle — that these ratios can entirely be expressed in terms of particle and quark mass parameters. For quark masses large compared to the spectator mass the Isgur-Wise relations follow. The corresponding amplitudes for heavy-to-light transitions show a very similar structure. In particular, the  $F_0$  and  $A_1$  form factors behave again differently from the  $F_1$ ,  $A_2$ ,  $V$  and  $T_1$  form factors.

The subject of this talk concerns exclusive semileptonic and radiative decays of heavy mesons. These decays play an outstanding role for the determination of the parameters of the standard model, in particular the quark-mixing parameters. They are also an important source of information about the still badly understood QCD dynamics in the confinement region. The dynamical content of the corresponding amplitudes is contained in Lorentz-invariant form factors. The calculation of these form factors requires a non-perturbative treatment. Many theoretical tools have been applied for their determination: quark models, QCD sum rules, and lattice calculations. The machinery of sum rules and lattice calculations has the advantage to be directly based on the QCD Lagrangian, but is quite involved. Quark models, on the other hand, are less directly connected with the QCD Lagrangian, but give a vivid picture of what is going on and allow an easy application to different processes and quite different kinematical regions. However, models presented so far lacked full relativistic covariance with respect to quark spins, and it is the spin structure which determines the ratio of different form factors we are concerned with here. The knowledge of these ratios is of paramount importance for the analysis of experimentally measured decays.

The problem of the ratios of different form factors is solved in the formal limit where initial and final quark masses are taken to be infinite. For instance, for  $m_c \rightarrow \infty, m_b \rightarrow \infty$  the two form factors ( $F_1, F_0$ ) describing  $B \rightarrow D$  transitions and the 4 form factors ( $V, A_0, A_1, A_2$ ) describing  $B \rightarrow D^*$  transitions are all related [1],[2]:

$$\begin{aligned} F_1 &= V = A_0 = A_2 \\ A_1 &= F_0 = (1 - q^2/(m_B + m_D)^2)F_1. \end{aligned} \quad (1)$$

Denoting initial and final meson masses by  $m_I$  and  $m_F$ , respectively, the Isgur-Wise function  $\xi(y)$  connected to  $F_1$  by

$$F_1 = \frac{1}{2} \sqrt{\frac{m_I}{m_F}} \left( 1 + \frac{m_F}{m_I} \right) \xi(y)_{Isgur-Wise} \quad (2)$$

depends in this limit on the product of the 4-velocities  $v_I$  and  $v_F$  only

$$y = v_F \cdot v_I = \frac{m_I^2 + m_F^2 - q^2}{2m_I m_F}. \quad (3)$$

The prediction of the corrections to the simple relations (1) for the case of finite physical quark masses are a challenge to quark models. The calculation of form factors is even more challenging for heavy-to-light transitions such as

$$\bar{B}^0 \rightarrow \pi^+ e^- \bar{\nu}_e, \quad \bar{B}^0 \rightarrow \rho^+ e^- \bar{\nu}_e. \quad (4)$$

The flavor and spin symmetries necessary to derive (1) do not apply in this case. Only the scaling

property of the matrix elements with respect to the mass  $m_I$  can be used. It connects, for example,  $B \rightarrow F$  with  $D \rightarrow F$  transitions. But this is of little help since the application is restricted to a limited kinematic region near  $q^2 = q_{max}^2$  and holds strictly only in the limit of large masses.

In the following I will use a fully relativistic quark model approach to get form factor relations. These relations follow without a detailed knowledge about quark model wave functions and should, therefore, be valid in the framework of a large class of dynamical models. They depend on mass parameters only and will reduce to eq. (1) in the limit of large masses of the active quarks.

Let us consider the 4-momentum of an initial meson ( $I$ ) with velocity  $v_I$  and divide it into the momenta of the active quark ( $i$ ) in this meson and the spectator ( $sp$ )

$$\begin{aligned} P^I &= p_i^I + p_{sp}^I \\ p_i^I &= \epsilon_i^I v_I + k_I, \quad p_{sp}^I = \epsilon_{sp}^I v_I - k_I \\ \epsilon_i^I + \epsilon_{sp}^I &= m_I. \end{aligned} \quad (5)$$

The dynamics of the bound state will be described by a momentum space wave function  $f_I(k_I, v_I)$ . Its peak is taken to be at  $k_I = 0$  defining, thereby, the splitting of the meson mass into the two constituent masses. Choosing the simplest internal  $S$ -wave structure, the wave function for the decaying pseudoscalar meson has the form (in coordinate space)

$$\begin{aligned} \psi_I(x_1, x_2) &= \frac{1}{N_I^{1/2}} \int d^4 k_I f_I(k_I^2, v_I \cdot k_I) \times \\ &(\not{p}_i^I + m_i) \gamma_5 (m_{sp} - \not{p}_{sp}^I) e^{-ip_i^I x_1} e^{-ip_{sp}^I x_2}. \end{aligned} \quad (6)$$

$\psi_I$  is a  $4 \times 4$  matrix. The mass values  $m_i, m_{sp}$  appearing in the propagators are not necessarily identical with the constituent masses  $\epsilon_i^I, \epsilon_{sp}^I$ . Expressions analog to (5,6) hold for the final particle (denoted by  $F$ ) emitted in the decay process. If it is a vector particle — a  $\rho$  meson for instance — the Dirac matrix  $\gamma_5$  has to be replaced by the polarisation matrix  $\not{\eta}$  of this particle.

The  $I \rightarrow F$  transition amplitude is obtained from the product  $\psi_I \bar{\psi}_F$ . The integration over the space-time coordinate  $x_2$  of the spectator particle leads to momentum conservation of this particle

$p_{sp}^F = p_{sp}^I$  correlating thereby the momentum  $k_F$  with  $k_I$ :

$$k_F - \epsilon_{sp}^F v_F = k_I - \epsilon_{sp}^I v_I \equiv K. \quad (7)$$

The off-shell quark momenta can thus be written

$$\begin{aligned} p_i^I(K) &= m_I v_I + K, \quad p_{sp}^I = p_{sp}^F = -K, \\ p_f^F(K) &= m_F v_F + K. \end{aligned} \quad (8)$$

The wave functions  $\psi_I, \psi_F$  contain the propagators of the active quarks and of the spectator. Because the weak current does not act on the spectator, the transition amplitude should contain this propagator only once as is evident from the corresponding triangle graph [3]. Thus, by inserting the inverse propagator of the spectator, the transition amplitude reads

$$\begin{aligned} &< F(v_F) | (\bar{q}_f(0) \Gamma q_i(0)) | I(v_I) > = \\ &\frac{(2\pi)^4}{(N_F N_I)^{1/2}} \int d^4 K f_F^*((K + \epsilon_{sp}^F v_F)^2, \\ &v_F \cdot K + \epsilon_{sp}^F) \cdot (m_{sp}^2 - p_{sp}^2) \cdot \\ &f_I((K + \epsilon_{sp}^I v_I)^2, v_I \cdot K + \epsilon_{sp}^I) \cdot J(K) \end{aligned} \quad (9)$$

with

$$\begin{aligned} J(K) &= \text{Spur} \left\{ \Gamma (\not{p}_i^I(K) + m_i^I) \gamma_5 \right. \\ &\left. (m_{sp} + \not{K}) \left( \not{\eta}^{\gamma_5} \right) (\not{p}_f^F + m_f^F) \right\}. \end{aligned} \quad (10)$$

In semileptonic decays  $\Gamma$  stands for  $\gamma_\mu(1 - \gamma_5)$ ; in the calculation of  $I \rightarrow F\gamma$  transitions it stands for  $\sigma_{\mu\nu} q^\nu(1 + \gamma_5)$  (apart from a global factor).

The initial and final wave functions have their maximum at  $k_F = k_I = 0$  with a width corresponding to the particle sizes. The integrand of the transition amplitude, on the other hand, has its maximum at values of  $k_B$  and  $k_F$  different from zero. One can expect this maximum to occur at  $k_I = \bar{k}_I, k_F = \bar{k}_F$  with

$$v_I \cdot \bar{k}_I = \frac{1}{2} (\epsilon_{sp}^I - \epsilon_{sp}^F) = -v_F \cdot \bar{k}_F. \quad (11)$$

The reason is that, together with (7,8), (11) implies small and equal off-shell values for all three propagators occurring in the transition matrix element, as well as average quark energies close to their constituent masses in the rest system of the

relevant particles: Because the average transverse components of  $p_{sp} = -\bar{K}$  vanish, one gets with  $\epsilon_{sp} = (\epsilon_{sp}^I + \epsilon_{sp}^F)/2$  from (7) and (11)

$$\bar{p}_{sp} = -\bar{K} = \epsilon_{sp} \frac{v_I + v_F}{1 + y} \quad (12)$$

and thus

$$\begin{aligned} \bar{p}_i^I \cdot v_I &= m_I - \epsilon_{sp} \\ \bar{p}_{sp} \cdot v_I &= \bar{p}_{sp} \cdot v_F = \epsilon_{sp} \\ \bar{p}_f^F \cdot v_F &= m_F - \epsilon_{sp} \\ (\bar{p}_i^I)^2 - (m_I - \epsilon_{sp})^2 &= (\bar{p}_{sp})^2 - \epsilon_{sp}^2 = \\ (\bar{p}_f^F)^2 - (m_F - \epsilon_{sp})^2 &= -\epsilon_{sp}^2 \frac{y-1}{y+1}, \\ \bar{k}_I^2 = \bar{k}_F^2 &= -\epsilon_{sp}^2 \frac{y-1}{y+1} + \left( \frac{\epsilon_{sp}^I - \epsilon_{sp}^F}{2} \right)^2. \end{aligned} \quad (13)$$

According to (12) the average space velocity of the spectator vanishes in the special coordinate system where  $\vec{v}_F = -\vec{v}_I$  as one would expect. In the following I will assume (12) to hold and to decisively determine the structure of the transition amplitude.

Considering the quark momenta (eq. (8)) in the physical region of the variable  $y = v_F \cdot v_I$  and taking  $\bar{K}$  for  $K$ , it is seen that at the maximum of the transition amplitude the decaying and emitted quarks carry essentially the same momenta as the mesons they are part of. This is especially true in heavy-to-heavy transitions where  $m_I, m_F \gg \epsilon_{sp}$ , but holds also in heavy-to-light decay processes at least for large values of  $y$  (i.e. low  $q^2$ -values) [4].

The covariant structure of the transition amplitude is obtained from the integral over  $J(K)$  in (9). For wave functions with a strong peak in momentum space and a width of order of the constituent mass of a light quark one may replace  $J(K)$  by  $J(\bar{K})$ . This replacement saves us from an unfruitful discussion of specific wave functions or propagators in the confinement region which cannot be reliably calculated at present. But it is certainly an approximation which holds good only for strongly peaked and otherwise smooth wave functions.

It is now a straightforward task to decompose

$$J(\bar{K}) = \text{Spur}\{\Gamma(m_I \not{v}_I + \bar{K} + m_i^I) \gamma_5 \cdot$$

$$(m_{sp} + \bar{K}) \left( \frac{\gamma_5}{\not{v}_F^*} \right) (m_F \not{v}_F + \bar{K} + m_f^F) \} \quad (14)$$

in terms of covariant expressions and to extract the corresponding form factors. There remains, of course, an undetermined function of the variable  $y$  multiplying the form factors. The result can be written in the form

$$\begin{aligned} F_1 &= \rho^{FI}(y)(1 + \zeta_{F1}^{FI}(y)) \\ F_0 &= \rho^{FI}(y)(1 - \frac{q^2}{(m_I + m_F)^2})(1 + \zeta_{F0}^{FI}(y)) \\ V &= \rho^{FI}(y)(1 + \zeta_V^{FI}(y)) \\ A_1 &= \rho^{FI}(y)(1 - \frac{q^2}{(m_I + m_F)^2})(1 + \zeta_{A1}^{FI}(y)) \\ A_2 &= \rho^{FI}(y)(1 + \zeta_{A2}^{FI}(y)) \\ A_0 &= \rho^{FI}(y)(1 + \zeta_{A0}^{FI}(y)) \\ T_1 &= \rho^{FI}(y)(1 + \zeta_{T1}^{FI}(y)). \end{aligned} \quad (15)$$

Here  $T_1$  is defined as the form factor relevant for radiative decays:

$$\begin{aligned} < F^* | (\bar{q}_f \sigma_{\mu\nu} (1 + \gamma_5) q^\nu q_i) | I > = \\ &\epsilon_{\mu\nu\lambda\rho} \eta_F^{*\nu} P_I^\lambda P_F^\rho 2T_1 - \\ &i(\eta_\mu^* (m_I^2 - m_F^2) - (\eta^* \cdot P^I)(P^I + P^F)_\mu) T_2 - \\ &i(\eta^* \cdot P^I) \cdot \\ &\left( (P^I - P^F)_\mu - \frac{q^2}{m_I^2 - m_F^2} (P^I + P^F)_\mu \right) T_3, \\ &T_2(q^2 = 0) = T_1(q^2 = 0), \quad \epsilon_{0123} = 1 \end{aligned} \quad (16)$$

The functions  $\zeta^{FI}(y)$  depend on dimensionless combinations of the masses contained in (14). They all vanish in the limit

$$m_i, m_I, m_f, m_F \gg m_{sp}, \epsilon_{sp}. \quad (17)$$

Thus, (15) contains the Isgur-Wise result eq. (1).

For the general case in which (17) does not hold, the functions  $\zeta^{FI}(y)$  could be written down but are too long to be displayed here.

The constituent quark picture suggests to use in these expressions (for a light quark spectator)

$$m_i = m_I - \epsilon_{sp}, \quad m_f = m_F - \epsilon_{sp}, \quad m_{sp} = \epsilon_{sp} \quad (18)$$

Then, the functions  $\zeta^{FI}(y)$  depend, besides upon  $y$  and the particle mass ratio  $m_F/m_I$ , upon the

value of  $\delta_F = \epsilon_{sp}/m_F$  only.<sup>1</sup> Since in  $B \rightarrow D^*, B \rightarrow \rho$  and  $D \rightarrow K^*, D \rightarrow \rho$  transitions  $\epsilon_{sp}^I \approx \epsilon_{sp}^F \approx 0.35$  GeV appears to be a reasonable value, all  $\zeta^{FI}$  functions are predictable using the corresponding values for  $\delta_F$ . As an example the functions  $1 + \zeta^{\rho B}(y)$  for the semi-leptonic  $B \rightarrow \rho$  transitions and the function  $1 + \zeta_{T_1}^{\rho B}$  are plotted in Fig. 1. ( $\zeta_{T_1}$  turned out to be identical to  $\zeta_{F_1}$  independent of the assumption (18)). Because the

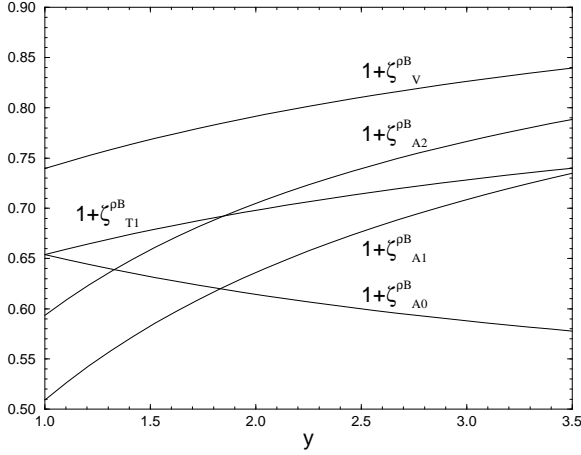


Figure 1. The functions  $1 + \zeta^{\rho B}$  for semileptonic and radiative  $B \rightarrow \rho$  decays. Quark masses according to Eq. (18) and  $\epsilon_{sp} = 0.35$  GeV.

curves for  $\zeta_V, \zeta_{A_1}$  and  $\zeta_{A_2}$  run similarly it is seen from (15) that the form factor  $A_1$  differs in its  $q^2$  dependence from the  $V$  and  $A_2$  form factors, like it is the case in the heavy quark limit. It is a consequence of relativistic covariance [5], [4].

In Fig. 2 the longitudinal polarization of the  $\rho$ -meson is shown using again (18) and  $\epsilon_{sp} = 0.35$  GeV. For comparison, the polarization without mass correction (i.e. with  $\zeta^{\rho B} = 0$ ) is also plotted.

<sup>1</sup>The use of current mass values, for instance  $m_b \simeq 4.8$  GeV,  $m_u \approx 0$  is also conceivable. Choosing in addition  $m_{sp} \approx 0$ , eq. (11) of ref. [4] can be rederived.

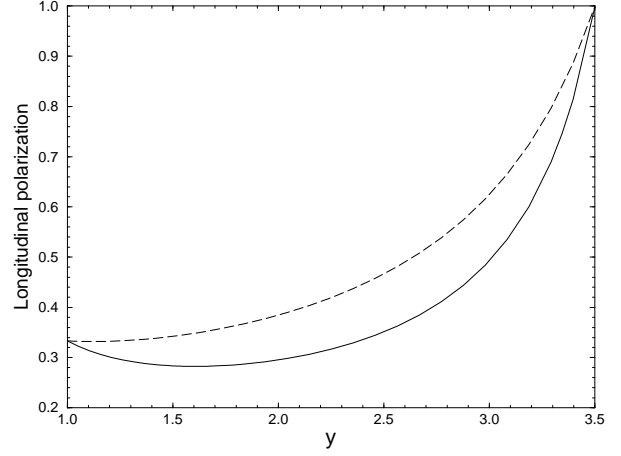


Figure 2. Longitudinal polarization of the  $\rho$ -meson in semileptonic  $B \rightarrow \rho$  transitions. Full line: quark masses according to Eq. (18) and  $\epsilon_{sp} = 0.35$  GeV. Dashed line: without mass corrections ( $\zeta^{\rho B} = 0$ ).

Because of the lack of spin symmetry in the light sector a relation between, for instance, the  $B \rightarrow \rho$  and  $B \rightarrow \pi$  form factors cannot be obtained from (15) even though the functions  $\zeta^{FI}$  are known. The factor  $\rho^{FI}(y)$  depends on the particle masses and on the internal structures of the initial and final particles. The explicit dependence on the particle masses can be taken care of, however, by setting

$$\rho^{FI} = \frac{1}{2} \sqrt{\frac{m_I}{m_F}} \left(1 + \frac{m_F}{m_I}\right) \xi^{FI}(y). \quad (19)$$

The form factors as obtained from (15) and (19) have now the correct scaling property for fixed  $y$ . Furthermore, for large masses of the active quarks  $\xi^{FI}(y)$  defined by (19) turns into the normalized Isgur-Wise function

$$\xi^{FI}(y) \rightarrow \xi_{Isgur-Wise}(y). \quad (20)$$

For phenomenological applications we can specify  $\xi^{FI}(y)$  further:

$$\xi^{FI}(y) = \sqrt{\frac{2}{y+1}} \left( \frac{1}{2} + \frac{1}{1+y} \right) g^{FI}(\bar{k}_I^2(y)), \quad (21)$$

$$\bar{k}_I^2(y) = \bar{k}_F^2(y) = -\epsilon_{sp}^2 \frac{y-1}{y+1} + \left( \frac{\epsilon_{sp}^I - \epsilon_{sp}^F}{2} \right)^2.$$

The first factor in (21) is necessary to give mass independence of the form factors in the limit  $m_I/m_F \rightarrow \infty$  at fixed  $q^2$ . The second factor is obtained from  $J(\bar{K})$  by setting  $m_{sp} = \epsilon_{sp}$ . It was divided out in defining the Isgur-Wise limit. The function  $g^{FI}$  depends on the variable  $\bar{k}_I^2 = \bar{k}_F^2$  obtained in (13) and is a increasing function of this variable.

As an illustration of the usefulness of (21) one may take a simple dipole formula which contains then just one parameter:

$$g^{FI} = \left( 1 - x^{FI} \bar{k}_I^2(y)/m_{sp}^2 \right)^{-2}. \quad (22)$$

$g^{FI}$ , or in the case of eq. (22), the parameter  $x^{FI}$  depends on the internal structure of initial and final states. Only in a hypothetical world where the internal meson wave functions are identical,  $g^{FI}$  would be process-independent and  $\xi^{FI}(y) = \xi(y)$  a truly universal function describing a large number of form factors. The difference from this hypothetical world seems, however, not to be a drastic one: Taking  $x = 0.5$ , the eqs. (21,22) provide for a reasonable Isgur-Wise function for  $B \rightarrow D^{(*)}$  decays. Together with (15), (19) one gets for the branching ratio  $BR(B \rightarrow D^{*} e^{-} \bar{\nu}_e) \approx 7\%$  consistent with the experimental value [6]. The same equations also lead to  $T_1(q^2 = 0) \approx 0.36$  for the  $B \rightarrow K^{*} \gamma$  process, a value also obtained in QCD sum rule estimates [7]. Applied to the semileptonic  $D \rightarrow K^{*}$  decay I obtain the branching ratio  $BR(D^0 \rightarrow \bar{K}^{*} e^{+} \nu) \approx 2\%$  also in accord with the data [6]. For the semileptonic  $\bar{B}^0 \rightarrow \rho^{+} e^{-} \bar{\nu}_e$  transition the form factors  $V^{\rho B}, A_1^{\rho B}, A_2^{\rho B}, A_0^{\rho B}$  at  $q^2 = 0$  turn out to be 0.37, 0.32, 0.35, 0.26, respectively, in accord with QCD sum rule results [7] and earlier estimates [8]. The differential branching ratio for the  $B \rightarrow \rho e^{-} \bar{\nu}$  transition is plotted in Fig. 3 - taking for the  $B$ -meson lifetime 1.5 psec and dividing by  $|V_{ub}|^2$ . Integrating it one finds the branching ratio  $21.1|V_{ub}|^2$ . (The transitions to the light pseudoscalars  $\pi$  and  $K$  require a more detailed treatment because the pole position is of greater importance; (18) is not applicable and  $\epsilon_{sp}^I \neq \epsilon_{sp}^F$ .)

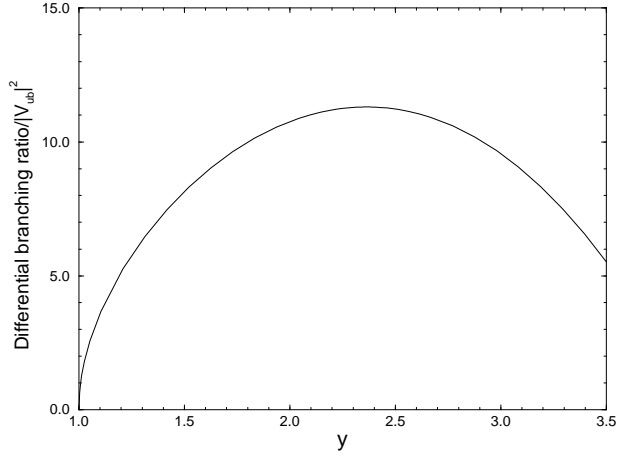


Figure 3. Differential branching ratio/ $|V_{ub}|^2$  for the semileptonic  $B \rightarrow \rho$  transition using  $\xi(y)$  as described in the text.

It should be clear that the “results” obtained from the Ansatz (22) and by taking an unjustified universal value for the parameter  $x^{FI}$  serve as an illustration only and are not based on a detailed analysis.

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